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THE TWO-DIMENSIONAL VORTEX MOTIONS OF A VISCOUS INCOMPRESSIBLE FLUID†

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A generalization of the Helmholtz theorems concerning the conservation of vortex lines and the intensity of vortex tubes, which are well known in the hydrodynamics of a non-viscous fluid, and of Kelvin's theorem on the conservation of the circulation of the velocity along a closed fluid contour is presented for the case of unsteady planar and axisymmetrical continuous flows of a viscous incompressible fluid.

PLANAR and axisymmetrical unsteady flows of a viscous incompressible fluid are studied on the assumption that the coefficient of kinematic viscosity ν is constant and that there are no external mass forces. Such flows are described by the well-known Navier–Stokes equations of motion, transformed to the Gromeka–Lamb form:

$$\frac{\partial \mathbf{v}}{\partial t}-\mathbf{v}\times\mathbf{\Omega}=\nabla\left(\frac{v^2}{2}+\frac{p}{\rho}\right)-\nu\,\mathrm{rot}\,\mathbf{\Omega},\quad \mathrm{div}\,\mathbf{v}=0 \tag{1}$$

where $\mathbf{v}$ is the velocity, p is the pressure, ρ is the density and $\Omega = \mathrm{rot}\,\mathbf{v}$.

Noting that Eqs (1) also hold in the case of spatial slows, let us consider the case of planar motions. In this case it has been previously shown that $\mathrm{rot}\,\Omega = \mathrm{rot}(\Omega\mathbf{k}) = -\Omega\times\nabla\ln\Omega$, where $\mathbf{k}$ is the unit vector of the z-axis along which the vorticity Ω is directed. This enables one to reduce (1) to the quasibarotropic form [1]

$$\frac{\partial \mathbf{v}}{\partial t}-\mathbf{u}\times\mathbf{\Omega}=-\nabla\left(\frac{v^2}{2}+\frac{p}{\rho}\right),\quad \mathbf{u}=\mathbf{v}-\nu\nabla\ln\mathbf{\Omega} \tag{2}$$

Here $\mathbf{u}$ is the generalized velocity [2] which reduces to the conventional velocity $\mathbf{v}$ of a fluid when there is no viscosity. Next, by carrying out the operation of rotation on (2), it can be transformed to the Friedman equation [3]

$$\mathrm{helm}\,\mathbf{\Omega}=\frac{D\mathbf{\Omega}}{Dt}-(\mathbf{\Omega}\nabla)\mathbf{u}+\mathbf{\Omega}\,\mathrm{div}\,\mathbf{u}=0\quad\left(\frac{D}{Dt}=\frac{\partial}{\partial t}+(\mathbf{u}\nabla)\right) \tag{3}$$

in the vectors Ω and $\mathbf{u}$.

However, according to Friedman's theorem [3], in the unsteady case relationship (3) is a necessary and sufficient condition for the conservation in time of the vector lines and intensities of the vector tubes for Ω as they are continuously displaced at a velocity $\mathbf{u}$. It therefore follows [3] from (3) that the two generalized Helmholtz theorems concerning the conservation of vortex lines and the intensities of vortex tubes are satisfied during their continuous displacement by a viscous incompressible fluid at a velocity $\mathbf{u}$.

In the special case of steady-state planar flows of a viscous fluid, it has previously been deduced [1] that vorticity is transported along vector y lines. It is important to note that the Bernoulli integral has also been found for these flows [1].

We will now show that it is possible to generalize Kelvin's theorem, concerning the conservation of the circulation of the velocity $\mathbf{v}$ along an arbitrary closed fluid contour C, if it is displaced at a velocity $\mathbf{u}$. Actually, by using Stokes' formula, we get

$$\oint_C \mathbf{v}\,dC=\iint_S \mathbf{\Omega}\,dS$$

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where S is the surface bounded by the contour C. Next, on taking the derivative D/Dt of this equality and differentiating on the right-hand side using the well-known formula of vector analysis [4], where the surface S continuously moves at a velocity $\mathbf{u}$, we find the required result

$$\frac{D}{Dt}\oint_C \mathbf{v}\, dC = \frac{D}{Dt}\iint_S \mathbf{\Omega}\, d\mathbf{S} = \iint_S \left[\frac{\partial \mathbf{\Omega}}{\partial t} + \mathbf{u}\,\mathrm{div}\,\mathbf{\Omega} + \mathrm{rot}(\mathbf{\Omega}\times\mathbf{u})\right] d\mathbf{S} = 0$$

where Eq. (3) has been used and account has been taken of the fact that $\mathrm{div}\,\Omega = 0$.

Similar results are also obtained in the axially symmetric case with cylindrical coordinates r, θ, z. For instance, it can be shown that [5]

$$\mathrm{rot}\,\mathbf{\Omega} = -\mathbf{\Omega}\times\nabla\ln|\Omega r| \qquad \left(\mathbf{\Omega} \equiv \mathrm{rot}\,\mathbf{v} \equiv \Omega\mathbf{e}_\theta,\ \nabla \equiv \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_z\frac{\partial}{\partial z}\right)$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{r}_z$ are the unit vectors along the coorindate axes. Hence, if one introduces a generalized velocity vector in the form [5] $\mathbf{u}' = \mathbf{v} - \nu\nabla\ln|\Omega r|$ and omits the prime here Friedman's equation remains precisely the same in form.

Consequently, the above-mentioned Holmholtz theorems and Kelvin's theorem also hold in the case of axisymmetrical unsteady flows of a viscous incompressible fluid if the vortex lines, the tubes and the contour C are continuously displaced by the fluid at a velocity $\mathbf{u}'$. In the steady-state axisymmetrical case, conclusions have previously been drawn [5] regarding the transport of vorticity and the Bernoulli integral along $\mathbf{u}'$-lines.

In concluding, we point out that the Holmholtz and Kelvin theorems have previously been extended [6–8] to the non-isoentropic flows of a viscous fluid and to adiabatic flows of a magnetized conducting fluid. We further note that all of the results of this paper can readily be generalized when account is taken of potential mass forces.

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